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Some ordered fixed point results and the property (P)

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ABSTRACT

In 2010, Kadelburg et al. ([7]) by providing an example showed that a contraction in an ordered metric space is not necessarily a contraction (in the classical sense). Thus fixed point results in ordered metric spaces are generalizations of ones in metric spaces in a sense. In this paper, we give some ordered fixed point results for convex contractions and special mappings which satisfy some contraction conditions. Also, we give some results concerning the property (P).

1. Introduction

The notion of metric spaces was introduced in 1906 by Maurice Frechet. Since then, many researchers have exploited this notion to define various concepts, using different views and ideas. One of the important notions is that of ordered metric spaces. We say that $(X, d, \preceq)$ is an ordered metric space whenever $\preceq$ is an order on X and (X, d) is a metric space. Although these spaces have been considered by many authors recently and even though ordered metric spaces were introduced and studied a few years ago, there are some old works on these spaces. For example, Vandergraft reviewed the Newton method for convex operators on partially ordered spaces in 1967 [1]. Also, Wolk reviewed continuous convergence on partially ordered spaces in 1975 [2]. Later, Sun and Sun started ordered fixed point theory in 1989 [3] and after some years it was continued by Agarwal et al. [4]. Also, Wanka published a paper concerning approximation theory in ordered spaces in 1996 [5]. In 2010, Altun et al. [6] and Kadelburg et al. [7] proved some fixed point and common fixed point theorems on ordered metric spaces by using a cone. In recent years, ordered fixed point theory has been considered by many authors (see, for example, [8–33]).

Rhoades defined the property (P) on metric spaces in his works [34–36]. Denote, as usual, by $F(T)$ the set of fixed points of the mapping $T : X \to X$. We say that a self-map T has the property (P) whenever $F(T) = F(T^n)$ for all $n \geq 1$, that is, it has no periodic points. Note that $F(T) \subseteq F(T^n)$ for all $n \geq 1$. Recently, two interesting papers have appeared on the property (P) [37,38]. More recently, Alghamdi et al. have studied convex contraction and two-sided convex contraction mappings [39], which were introduced by Istratescu [40] in 1982. We use these notions to obtain some results. In this paper, we give some ordered fixed point results for convex contractions and special mappings which satisfy some contraction conditions and are not necessarily continuous. Also, we give some results concerning the property (P).

2. The main results

Let $(X, \preceq)$ be a partially ordered set. We define $X_{\preceq} = \{(x, y) \in X \times X : x \preceq y \text{ or } y \preceq x\}$. Also, we say that a self-map $T : X \to X$ is orbitally continuous at x whenever for each sequence $\{n(i)\}_{i\geq 1}$ with $T^{n(i)}x \to a$ for some $a \in X$, we have

$$T^{n(i)+1}x \to Ta.$$

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Here, $T^{m+1} = T(T^m)$. Finally, we define the orbit of T at x by

$$O(x, \infty) := \{x, Tx, T^2x, \dots, T^nx, \dots\}$$

and we say that T has the strongly comparable property whenever $(T^{n-1}y, T^ny) \in X_{\leq}$ for all $n \geq 1$ and $m \geq 2$, where $y \in F(T^m)$.

Theorem 2.1. *Let $(X, d, \leq)$ be a complete ordered metric space, $\lambda \in (0, 1)$ and T a self-map on X satisfying the condition*

$$\min\{d^2(Tx, Ty), d(x, y)d(Tx, Ty), d^2(y, Ty)\} - \min\{d^2(x, Tx), d(y, Ty)d(x, Ty), d^2(y, Tx)\} \leq \lambda\, d(x, Tx)d(y, Ty)$$

for all $x, y \in X_{\leq}$. If there exists $x_0 \in X$ such that $(T^{n-1}x_0, T^nx_0) \in X_{\leq}$ for all $n \geq 1$ and T is orbitally continuous at x_0, then T has a fixed point. Moreover, if T has the strongly comparable property, then T has the property (P).

Proof. Define $x_{n+1} = Tx_n = T^{n+1}x_0$ for all $n \geq 0$. If $x_{n_0} = x_{n_0-1}$ for some natural number n_0, then $x_n = x_{n_0}$ for all $n \geq n_0$ and x_{n_0} is a fixed point of T. Suppose that $x_n \neq x_{n-1}$ for all $n \geq 1$. Now for each $n \geq 1$, by using the assumption, we can put $x = x_{n-1}$ and $y = x_n$ in the condition. Thus, we obtain

$$\min\{d^2(x_n, x_{n+1}), d(x_{n-1}, x_n)d(x_n, x_{n+1})\} \leq \lambda\, d(x_{n-1}, x_n)d(x_n, x_{n+1}).$$

Since $\lambda < 1$, $\min\{d^2(x_n, x_{n+1}), d(x_{n-1}, x_n)d(x_n, x_{n+1})\} = d^2(x_n, x_{n+1})$. Hence,

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n).$$

By continuing this process we obtain $d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1)$ for all $n \geq 1$. Thus, for each natural number k, we have

$$d(x_n, x_{n+k}) \leq \sum_{i=n}^{n+k-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{n+k-1} \lambda^i d(x_0, x_1) \leq \frac{\lambda^n}{1-\lambda} d(x_0, x_1).$$

Therefore, $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, there exists $u \in X$ such that $x_n \to u$. Since T is orbitally continuous, $x_{n+1} = Tx_n \to Tu$. This implies that $Tu = u$. Now, we prove that T has the property (P). Let $n \geq 2$ be given and $v \in F(T^n)$. Since T has the strongly comparable property, we can put $x = T^{n-1}v$ and $y = T^nv$ in the condition. Thus, we obtain

$$\min\{d^2(T^nv, T^{n+1}v), d(T^{n-1}v, T^nv)d(T^nv, T^{n+1}v)\} \leq \lambda\, d(T^{n-1}v, T^nv)d(T^nv, T^{n+1}v).$$

Thus, $\min\{d^2(v, Tv), d(T^{n-1}v, v)d(v, Tv)\} \leq \lambda\, d(T^{n-1}v, v)d(v, Tv)$ and so two cases arise.

Case I. $d^2(v, Tv) \leq \lambda\, d(T^{n-1}v, v)d(v, Tv)$.

We claim that $d(v, Tv) = 0$. If $d(v, Tv) > 0$, then $d(v, Tv) = d(T^nv, T^{n+1}v) \leq \lambda\, d(T^{n-1}v, T^nv)$. By putting $x = T^{n-2}v$ and $y = T^{n-1}v$ in the condition, we obtain

$$\min\{d^2(T^{n-1}v, T^nv), d(T^{n-2}v, T^{n-1}v)d(T^{n-1}v, T^nv)\} \leq \lambda\, d(T^{n-2}v, T^{n-1}v)d(T^{n-1}v, T^nv).$$

Let $d^2(T^{n-1}v, T^nv) \leq \lambda\, d(T^{n-2}v, T^{n-1}v)d(T^{n-1}v, T^nv)$. If $d(T^{n-1}v, T^nv) = 0$, then $T^{n-1}v = v$ and so $v = T^nv = Tv$. If $d(T^{n-1}v, T^nv) > 0$, then $d(T^{n-1}v, T^nv) \leq \lambda\, d(T^{n-2}v, T^{n-1}v)$. Now, let

$$d(T^{n-2}v, T^{n-1}v)d(T^{n-1}v, T^nv) \leq \lambda\, d(T^{n-2}v, T^{n-1}v)d(T^{n-1}v, T^nv).$$

So we should have $d(T^{n-2}v, T^{n-1}v) = 0$ or $d(T^{n-1}v, T^nv) = 0$ (and so $v = Tv$), because if $d(T^{n-2}v, T^{n-1}v) > 0$ and $d(T^{n-1}v, T^nv) > 0$, then we get $\lambda \geq 1$ which is a contradiction. By continuing this process, we obtain

$$d(v, Tv) = d(T^nv, T^{n+1}v) \leq \lambda\, d(T^{n-1}v, T^nv) \leq \lambda^2\, d(T^{n-2}v, T^{n-1}v) \cdots \leq \lambda^n\, d(v, Tv)$$

which leads us to $\lambda \geq 1$ which is a contradiction. Therefore, in this case we have $d(v, Tv) = 0$ and so $Tv = v$.

Case II. $d(T^{n-1}v, v)d(v, Tv) \leq \lambda\, d(T^{n-1}v, v)d(v, Tv)$.

In this case we should have $d(T^{n-1}v, v) = 0$ or $d(v, Tv) = 0$ (and so $v = Tv$). In fact, if $d(T^{n-1}v, v) > 0$ and $d(v, Tv) > 0$, then $\lambda \geq 1$ which is a contradiction. Thus, we have the consequence that $F(T^n) \subseteq F(T)$. Therefore, T has the property (P). □

The following example shows that there are nonlinear and discontinuous mappings which satisfy the condition of Theorem 2.1.

Example 2.1. Let $X = [0, \infty)$, $d(x, y) = |x - y|$ and T be a self-map on X defined by $Tx = 0$ whenever $0 \leq x \leq 10$, $Tx = x - 10$ whenever $10 \leq x \leq 11$ and $Tx = 1.1$ whenever $x > 11$. Then, on putting $\lambda = \frac{1}{2}$, T satisfies the condition of Theorem 2.1.

Theorem 2.2. *Let $(X, d, \leq)$ be a complete ordered metric space, $\lambda \in (0, 1)$, $a \geq 0$ and T a self-map on X satisfying the condition*

$$\min\left\{d(Tx,Ty), d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)[1+d(y,Ty)]}{1+d(x,y)}, \frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)},\right.$$
$$\left.\frac{\min\{d^2(Tx,Ty), d^2(x,Tx), d^2(y,Ty)\}}{d(x,y)}\right\} - a\min\left\{\frac{d^2(x,Tx)[1+d^2(y,Ty)]}{1+d^2(x,y)}, \frac{d^2(y,Ty)[1+d^2(x,Tx)]}{1+d^2(x,y)},\right.$$
$$\left. d(x,Tx), d(y,Tx)\right\} \leq \lambda \max\{d(x,y), d(x,Tx)\}$$

for all $x, y \in X_{\leq}$. If there exists $x_0 \in X$ such that $(T^{n-1}x_0, T^n x_0) \in X_{\leq}$ for all $n \geq 1$ and T is orbitally continuous at x_0, then T has a fixed point. Moreover, if T has the strongly comparable property, then T has the property (P).

Proof. Define $x_{n+1} = Tx_n = T^{n+1}x_0$ for all $n \geq 0$. If $x_{n_0} = x_{n_0-1}$ for some natural number n_0, then $x_n = x_{n_0}$ for all $n \geq n_0$ and x_{n_0} is a fixed point of T. Suppose that $x_n \neq x_{n-1}$ for all $n \geq 1$. Now, by using the assumption, we can put $x = x_0$ and $y = x_1 = Tx_0$ in the condition. Then, we obtain

$$\min\left\{d(Tx_0,T^2x_0), d(x_0,Tx_0), \frac{d(x_0,Tx_0)[1+d(Tx_0,T^2x_0)]}{1+d(x_0,Tx_0)}, \frac{\min\{d^2(Tx_0,T^2x_0), d^2(x_0,Tx_0)\}}{d(x_0,Tx_0)}\right\} \leq \lambda\, d(x_0,Tx_0).$$

If $d(x_0,Tx_0) \leq d(Tx_0,T^2x_0)$, then we deduce that $d(x_0,Tx_0) \leq \lambda\, d(x_0,Tx_0)$ and so $\lambda \geq 1$ which is a contradiction. This implies that $d(Tx_0,T^2x_0) \leq d(x_0,Tx_0)$. Thus,

$$\min\left\{d(Tx_0,T^2x_0), d(x_0,Tx_0), \frac{d(x_0,Tx_0)[1+d(Tx_0,T^2x_0)]}{1+d(x_0,Tx_0)}, \frac{d^2(Tx_0,T^2x_0)}{d(x_0,Tx_0)}\right\} \leq \lambda\, d(x_0,Tx_0).$$

But it is easy to see that

$$\min\left\{d(Tx_0,T^2x_0), d(x_0,Tx_0), \frac{d(x_0,Tx_0)[1+d(Tx_0,T^2x_0)]}{1+d(x_0,Tx_0)}, \frac{d^2(Tx_0,T^2x_0)}{d(x_0,Tx_0)}\right\} = \frac{d^2(Tx_0,T^2x_0)}{d(x_0,Tx_0)}$$

and so $\frac{d^2(Tx_0,T^2x_0)}{d(x_0,Tx_0)} \leq \lambda\, d(x_0,Tx_0)$. Hence, $d(Tx_0,T^2x_0) \leq \sqrt{\lambda}\, d(x_0,Tx_0)$. Now, by continuing this process, for each $n \geq 1$ we obtain

$$d(T^nx_0,T^{n+1}x_0) \leq \sqrt{\lambda}\, d(T^{n-1}x_0,T^nx_0) \leq \cdots \leq \sqrt{\lambda^n}\, d(x_0,Tx_0).$$

Thus, for each natural number k we have

$$d(x_n,x_{n+k}) = d(T^nx_0,T^{n+k}x_0) \leq \sum_{i=n}^{n+k-1} \sqrt{\lambda^i}\, d(x_0,Tx_0) \leq \frac{\sqrt{\lambda^n}}{1-\sqrt{\lambda}} d(x_0,Tx_0).$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, there exists $u \in X$ such that $x_n \to u$. Since T is orbitally continuous, $Tx_n \to Tu$. Hence, u is a fixed point of T. Now, we prove that T has the property (P). Let $n \geq 2$ be given and $v \in F(T^n)$. If $d(v,Tv) = 0$, then $Tv = v$. Suppose that $d(v,Tv) > 0$. An argument similar to that in the first part of this proof shows that

$$\begin{aligned}\frac{d^2(Tv,T^2v)}{d(v,Tv)} &= \min\left\{d(Tv,T^2v), d(v,Tv), \frac{d(v,Tv)[1+d(Tv,T^2v)]}{1+d(v,Tv)}, \frac{d^2(Tv,T^2v)}{d(v,Tv)}\right\}\\ &= \min\left\{d(Tv,T^2v), d(v,Tv), \frac{d(v,Tv)[1+d(Tv,T^2v)]}{1+d(v,Tv)}, \frac{\min\{d^2(Tv,T^2v), d^2(v,Tv)\}}{d(v,Tv)}\right\}\\ &\leq \lambda\, d(v,Tv).\end{aligned}$$

Hence, $d(Tv,T^2v) \leq \sqrt{\lambda}\, d(v,Tv)$. Since T has the strongly comparable property, by using an argument similar to that in Theorem 2.1, we obtain

$$d(v,Tv) = d(T^nv,T^{n+1}v) \leq \sqrt{\lambda}\, d(T^{n-1}v,T^nv) \leq \cdots \leq \sqrt{\lambda^n}\, d(v,Tv).$$

This implies that $Tv = v$ because $\lambda < 1$. Thus, $F(T^n) \subseteq F(T)$. Therefore, T has the property (P). □

The following example shows that there are nonlinear and discontinuous mappings which satisfy the condition of Theorem 2.2.

Example 2.2. Let $X = [0, \infty)$, $d(x, y) = |x - y|$ and T be a self-map on X defined by $Tx = 0$ whenever $0 \le x \le 20$, $Tx = x - 20$ whenever $20 \le x \le 21$ and $Tx = 1.1$ whenever $x > 21$. Then, on putting $\lambda = \frac{1}{2}$ and $a = 1$, T satisfies the condition of Theorem 2.2.

The following result generalizes Proposition 1.1 of [34].

Theorem 2.3. *Let $(X, d, \le)$ be a complete ordered metric space, $b \in [0, 1)$, $c \ge 0$, m a nonnegative integer and T a self-map on X satisfying the condition*

$$d^2(T^{m+1}x, T^{m+2}y) \le b\, d(T^m x, T^{m+1}x)d(T^{m+1}y, T^{m+2}y) + c\, d(T^m x, T^{m+2}y)d(T^{m+1}y, T^{m+1}x)$$

for all $x, y \in X_\le$. Suppose that there exists $x_0 \in X$ such that $(T^{n-1}x_0, T^n x_0) \in X_\le$ for all $n \ge 1$. If T is orbitally continuous at x_0 or $m = 0$, then T has a fixed point. Moreover, T has a unique fixed point whenever $c < 1$. If T has the strongly comparable property, then T has the property (P).

Proof. Define $x_1 = T^{m+1}x_0$ and $x_{n+1} = Tx_n$ for all $n \ge 1$. Then

$$\begin{aligned} d^2(x_1, x_2) &= d^2(T^{m+1}x_0, T^{m+2}x_0) \le b\, d(T^m x_0, T^{m+1}x_0)d(T^{m+1}x_0, T^{m+2}x_0) \\ &\quad + c\, d(T^m x_0, T^{m+2}x_0)d(T^{m+1}x_0, T^{m+1}x_0) \\ &= b\, d(T^m x_0, T^{m+1}x_0)d(T^{m+1}x_0, T^{m+2}x_0) = b\, d(T^m x_0, T^{m+1}x_0)d(x_1, x_2). \end{aligned}$$

If $d(x_1, x_2) = 0$, then $Tx_1 = x_2 = x_1$ and so T has a fixed point. If $d(x_1, x_2) > 0$, then $d(x_1, x_2) \le bd(T^m x_0, x_1)$. Similarly, we have

$$\begin{aligned} d^2(x_2, x_3) &= d^2(T^{m+2}x_0, T^{m+3}x_0) \le b\, d(T^{m+1}x_0, T^{m+2}x_0)d(T^{m+2}x_0, T^{m+3}x_0) \\ &\quad + c\, d(T^{m+1}x_0, T^{m+3}x_0)d(T^{m+2}x_0, T^{m+2}x_0) \\ &= b\, d(T^{m+1}x_0, T^{m+2}x_0)d(T^{m+2}x_0, T^{m+3}x_0) = b\, d(x_1, x_2)d(x_2, x_3). \end{aligned}$$

If $d(x_2, x_3) = 0$, then $Tx_2 = x_3 = x_2$ and so T has a fixed point. If $d(x_2, x_3) > 0$, then $d(x_2, x_3) \le bd(x_1, x_2)$ and so $d(x_2, x_3) \le b^2 d(T^m x_0, x_1)$. By continuing this process we get that $d(x_n, x_{n+1}) \le b^n d(T^m x_0, x_1)$ for all $n \ge 1$. This implies that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, there exists $u \in X$ such that $x_n \to u$. If T is orbitally continuous, then $Tx_n \to Tu$. Hence, $Tu = u$.
If $m = 0$, then for each $n \ge 2$ we have

$$d^2(Tu, T^n x_0) \le b\, d(u, Tu)d(Tx_{n-2}, T^2 x_{n-2}) + c\, d(u, T^2 x_{n-2})d(Tx_{n-2}, Tu).$$

Since $x_n \to u$, we have $d^2(Tu, u) \le c\, d(u, u)d(u, Tu) = 0$ and so $Tu = u$. Now, we show that T has a unique fixed point whenever $c < 1$. Let u and v be fixed points of T. Then, we have

$$\begin{aligned} d^2(u, v) &= d^2(T^{m+1}u, T^{m+2}v) \le b\, d(T^m u, T^{m+1}u)d(T^{m+1}v, T^{m+2}v) + c\, d(T^m u, T^{m+2}v)d(T^{m+1}v, T^{m+1}u) \\ &= c\, d^2(u, v). \end{aligned}$$

Hence, $d(u, v) = 0$ because $c < 1$. Thus, $u = v$ and so T has a unique fixed point.
Finally, we prove that T has the property (P) whenever T has the strongly comparable property. Let $n \ge 2$ be given and $v \in F(T^n)$. We consider the following cases.

Case I. $m = 0$.
In this case, we have

$$\begin{aligned} d^2(v, Tv) &= d^2(T(T^{n-1}v), T^2(T^{n-1}v)) \le b\, d(T^{n-1}v, T^n v)d(T^n v, T^{n+1}v) + c\, d(T^{n-1}v, T^{n+1}v)d(T^n v, T^n v) \\ &= b\, d(T^{n-1}v, v)d(v, Tv). \end{aligned}$$

If $d(v, Tv) = 0$, then $Tv = v$. If $d(v, Tv) > 0$, then

$$d(T^n v, T^{n+1}v) \le bd(T^{n-1}v, T^n v).$$

By continuing the process and using an argument similar to that in Theorem 2.1, we obtain

$$\begin{aligned} d(v, Tv) &= d(T^n v, T^{n+1}v) \le bd(T^{n-1}v, T^n v) \\ &\le b^2 d(T^{n-2}v, T^{n-1}v) \cdots \le b^n d(v, Tv). \end{aligned}$$

Since $b < 1$, $Tv = v$.

Case II. $m \geq 1$ and $n > m$.
In this case, we have

$$\begin{aligned} d^2(v, Tv) &= d^2(T^{m+1}(T^{n-m-1}v), T^{m+2}(T^{n-m-1}v)) \\ &\leq b\, d(T^m(T^{n-m-1}v), T^{m+1}(T^{n-m-1}v))d(T^{m+1}(T^{n-m-1}v), T^{m+2}(T^{n-m-1}v)) \\ &\quad + c\, d(T^m(T^{n-m-1}v), T^{m+2}(T^{n-m-1}v))d(T^{m+1}(T^{n-m-1}v), T^{m+1}(T^{n-m-1}v)) \\ &= b\, d(T^{n-1}v, T^n v)d(T^n v, T^{n+1}v) + c\, d(T^{n-1}v, T^{n+1}v)d(T^n v, T^n v) \\ &= b\, d(T^{n-1}v, v)d(v, Tv). \end{aligned}$$

If $d(v, Tv) = 0$, then $Tv = v$. If $d(v, Tv) > 0$, then

$$d(T^n v, T^{n+1}v) \leq bd(T^{n-1}v, T^n v).$$

By continuing the process and using an argument similar to that in Theorem 2.1, we obtain

$$\begin{aligned} d(v, Tv) &= d(T^n v, T^{n+1}v) \leq bd(T^{n-1}v, T^n v) \\ &\leq b^2 d(T^{n-2}v, T^{n-1}v) \cdots \leq b^n d(v, Tv). \end{aligned}$$

Since $b < 1$, $Tv = v$.
Case III. $m \geq 1$ and $n \leq m$.
In this case, choose a natural number r and an integer number $0 \leq s < n$ such that $m + 1 = rn + s$. Then, we have $T^m(T^{n-s}v) = T^{n-1}v$, $T^{m+1}(T^{n-s}v) = v$ and so

$$\begin{aligned} d^2(v, Tv) &= d^2(T^{m+1}(T^{n-s}v), T^{m+2}(T^{n-s}v)) \\ &\leq b\, d(T^m(T^{n-s}v), T^{m+1}(T^{n-s}v))d(T^{m+1}(T^{n-s}v), T^{m+2}(T^{n-s}v)) \\ &\quad + c\, d(T^m(T^{n-s}v), T^{m+2}(T^{n-s}v))d(T^{m+1}(T^{n-s}v), T^{m+1}(T^{n-s}v)) \\ &= b\, d(T^{n-1}v, v)d(v, Tv). \end{aligned}$$

If $d(v, Tv) = 0$, then $Tv = v$. If $d(v, Tv) > 0$, then

$$d(T^n v, T^{n+1}v) \leq bd(T^{n-1}v, T^n v).$$

By continuing the process and using an argument similar to that in Theorem 2.1, we obtain

$$\begin{aligned} d(v, Tv) &= d(T^n v, T^{n+1}v) \leq bd(T^{n-1}v, T^n v) \\ &\leq b^2 d(T^{n-2}v, T^{n-1}v) \cdots \leq b^n d(v, Tv). \end{aligned}$$

Since $b < 1$, $Tv = v$. Thus, $F(T^n) \subseteq F(T)$. Therefore, T has the property (P). □

The following example shows that there are nonlinear and discontinuous mappings satisfying the condition of Theorem 2.3.

Example 2.3. Let $X = [0, \infty)$, $d(x, y) = |x - y|$ and T be a self-map on X defined by $Tx = 0$ whenever $0 \leq x \leq 100$, $Tx = x - 100$ whenever $100 \leq x \leq 100.1$ and $Tx = 0.15$ whenever $x > 100.1$. Then, on putting $m = 0$, $b = \frac{1}{4}$ and $c = \frac{1}{2}$, T satisfies the condition of Theorem 2.3.

The following examples show that there are some mappings satisfying the condition of Theorem 2.3 while not satisfying the condition of Proposition 1.1 of [34].

Example 2.4. Let $X = \{1, 6, 8\}$, $d(x, y) = |x - y|$, $\leq = \{(1, 1), (6, 6), (8, 8)\}$ and T be a self-map on X defined by $T1 = 6$, $T6 = 8$ and $T8 = 8$. Then, on putting $m = 0$, $x_0 = 8$, $b = \frac{1}{2}$ and $c = 0.9$, it is easy to see that T satisfies the condition of Theorem 2.3 while for $x = 1$ and $y = 8$ we have $49 = d^2(1, 8) > bd(1, 6)d(6, 8) + cd(1, 8)d(8, 6) = 17.6$. This implies that T does not satisfy the condition of Proposition 1.1 of [34].

Example 2.5. Let $X = \{1, 2, 3, 4, 5\}$, $\leq = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 3)\}$, $d(x, y) = |x - y|$ and T be a self-map on X defined by $T1 = 4$, $T2 = 3$, $T3 = 3$, $T4 = 4$ and $T5 = 5$. Then, on putting $m = 0$, $x_0 = 4$, $b = \frac{1}{2}$ and $c = 0.6$, it is easy to see that T satisfies the condition of Theorem 2.3 while for $x = 2$ and $y = 5$ we have $4 = d^2(3, 5) > bd(2, 3)d(5, 5) + cd(2, 5)d(5, 3) = 3.6$. This implies that T does not satisfy the condition of Proposition 1.1 of [34].

In [39], Alghamdi et al. have studied convex contraction and two-sided convex contraction mappings, which were introduced by Istratescu [40]. These notions were defined for continuous mappings. We change them here to the following.

Definition 2.6. Let (X, d) be a metric space and T a self-map on X. Then T is said to be a convex contraction if there exist $a, b \in (0, 1)$ with $a + b < 1$ such that

$$d(T^2x, T^2y) \leq a\, d(Tx, Ty) + b\, d(x, y)$$

for all $x, y \in X$. Also, T is said to be a convex contraction of order 2 if there exist $a_1, a_2, b_1, b_2 \in (0, 1)$ with $a_1+a_2+b_1+b_2 < 1$ such that

$$d(T^2x, T^2y) \leq a_1\, d(x, Tx) + a_2\, d(Tx, T^2x) + b_1\, d(y, Ty) + b_2\, d(Ty, T^2y)$$

for all $x, y \in X$.

It is known that there are convex contractions of order 2 which are not contractions on metric spaces [39,40]. Our next result generalizes a result of Istratescu [40] (see also [39]).

Theorem 2.4. *Let $(X, d, \leq)$ be a complete ordered metric space, $a, b \in (0, 1)$ with $a + b < 1$ and T an orbitally continuous self-map on X satisfying the condition*

$$d(T^2x, T^2y) \leq a\, d(Tx, Ty) + b\, d(x, y)$$

for all $x, y \in X_{\leq}$. If there exists $x_0 \in X$ such that $(T^{n-1}x_0, T^nx_0) \in X_{\leq}$ for all $n \geq 1$, then T has a unique fixed point. Also, $F(T) = F(T^2)$.

Proof. Define $x_n = T^nx_0$ for all $n \geq 1$, $\upsilon = d(Tx_0, T^2x_0) + d(x_0, Tx_0)$ and $\lambda = a + b$. Then by using a technique similar to that in the proof of Theorem 3 of [39], it is easy to see that $d(T^{m+1}x_0, T^mx_0) \leq 2\lambda^l\upsilon$, where $m = 2l$ or $m = 2l + 1$. Also, $d(T^mx_0, T^nx_0) \leq \frac{4\lambda^l}{1-\lambda}\upsilon$ for all $n > m$ and so $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists $u \in X$ such that $x_n \to u$. Since T is orbitally continuous, $Tx_n \to Tu$. Hence, we get $Tu = u$. Now, we show that T has a unique fixed point. Let y and z be fixed points of T. Then $d(y, z) = d(T^2y, T^2z) \leq a\, d(Ty, Tz) + b\, d(y, z) = (a + b)d(y, z)$. Since $a + b < 1$, we obtain the consequence that $y = z$. Now, we prove that $F(T) = F(T^2)$. Let $y \in F(T^2)$. Then, we have

$$d(Ty, y) = d(T^2Ty, T^2y) \leq a\, d(T^2y, Ty) + b\, d(Ty, y) = (a + b)d(y, Ty).$$

Since $a + b < 1$, we get $Ty = y$. This completes the proof. □

Like in Examples 2.1, 2.2 and 2.2, we can show that there are nonlinear and discontinuous mappings satisfying the condition of Theorem 2.4. The following examples show that there are some mappings satisfying the condition of Theorem 2.4 while not satisfying the condition of Theorem 1.2 of [40] (see also [39]). In 2011, Haghi et al. proved that some fixed point generalizations are not real generalizations [41]. But the following examples show that our results are real generalizations.

Example 2.7. Let $X = \{1, 3, 5\}$, $d(x, y) = |x - y|$, $\leq = \{(1, 1), (3, 3), (5, 5)\}$ and T be a self-map on X defined by $T1 = 3$, $T3 = 1$ and $T5 = 5$. Then, on putting $x_0 = 5$, $a = \frac{1}{2}$ and $b = \frac{1}{4}$, it is easy to see that T satisfies the condition of Theorem 2.4 while for $x = 1$ and $y = 3$ we have $2 = d(T^21, T^23) > ad(T1, T3) + bd(1, 3) = 1.5$. This implies that T does not satisfy the condition of Theorem 1.2 of [40].

Example 2.8. Let $X = \{1, 2, 3, 4, 5\}$, $\leq = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 3)\}$, $d(x, y) = |x - y|$ and T be a self-map on X defined by $T1 = 4$, $T2 = 3$, $T3 = 3$, $T4 = 4$ and $T5 = 5$. Then, on putting $a = \frac{1}{10}$, $x_0 = 4$ and $b = \frac{1}{2}$, it is easy to see that T satisfies the condition of Theorem 2.4 while for $x = 2$ and $y = 5$ we have $2 = d(3, 5) > ad(3, 5) + bd(2, 5) = 1.7$. This implies that T does not satisfy the condition of Theorem 1.2 of [40].

The next result generalizes Theorem 2.3 of [40] (see also [39]).

Theorem 2.5. *Let $(X, d, \leq)$ be a complete ordered metric space, $a_1, a_2, b_1, b_2 \in (0, 1)$ with $a_1 + a_2 + b_1 + b_2 < 1$ and T an orbitally continuous self-map on X satisfying the condition $d(T^2x, T^2y) \leq a_1\, d(x, Tx)+a_2\, d(Tx, T^2x)+b_1\, d(y, Ty)+b_2\, d(Ty, T^2y)$ for all $x, y \in X_{\leq}$. If there exists $x_0 \in X$ such that $(T^{n-1}x_0, T^nx_0) \in X_{\leq}$ for all $n \geq 1$, then T has a unique fixed point. Also, $F(T) = F(T^2)$.*

Proof. Define $x_n = T^nx_0$ for all $n \geq 1$, $\upsilon = d(Tx_0, T^2x_0) + d(x_0, Tx_0)$. Also, put $\lambda = a_1 + a_2 + b_1$ and $\beta = 1 - b_2$. First, note that

$$\begin{aligned} d(T^3x_0, T^2x_0) &\leq a_1\, d(x_0, Tx_0) + a_2\, d(Tx_0, T^2x_0) + b_1\, d(Tx_0, T^2x_0) + b_2\, d(T^3x_0, T^2x_0) \\ &\leq a_1\upsilon + (a_2 + b_1)\upsilon + b_2\, d(T^3x_0, T^2x_0). \end{aligned}$$

Hence, $d(T^3x_0, T^2x_0) \leq (\frac{\lambda}{\beta})\upsilon$. Now, by using the assumption, we can put $x = Tx_0$ and $y = T^2x_0$ in the condition. Thus, we obtain

$$\begin{aligned} d(T^3x_0, T^4x_0) &\leq a_1\, d(Tx_0, T^2x_0) + a_2\, d(T^2x_0, T^3x_0) + b_1\, d(T^2x_0, T^3x_0) + b_2\, d(T^3x_0, T^4x_0) \\ &\leq a_1\upsilon + (a_2 + b_1)\frac{a_1 + a_2 + b_1}{1 - b_2}\upsilon + b_2\, d(T^3x_0, T^4x_0). \end{aligned}$$

Hence, $d(T^3x_0, T^4x_0) \leq (\frac{\lambda}{\beta})v$. Similarly, we have

$$\begin{aligned} d(T^5x_0, T^4x_0) &\leq a_1\, d(T^2x_0, T^3x_0) + a_2\, d(T^3x_0, T^4x_0) + b_1\, d(T^3x_0, T^4x_0) + b_2\, d(T^5x_0, T^4x_0) \\ &\leq a_1 \frac{a_1 + a_2 + b_1}{1 - b_2} v + (a_2 + b_1) \frac{a_1 + a_2 + b_1}{1 - b_2} v + b_2\, d(T^5x_0, T^4x_0). \end{aligned}$$

Hence, $d(T^5x_0, T^4x_0) \leq (\frac{\lambda}{\beta})^2 v$. Also, by using the assumption and putting $x = T^3x_0$ and $y = T^4x_0$ in the condition, we obtain

$$\begin{aligned} d(T^5x_0, T^6x_0) &\leq a_1\, d(T^3x_0, T^4x_0) + a_2\, d(T^4x_0, T^5x_0) + b_1\, d(T^4x_0, T^5x_0) + b_2\, d(T^5x_0, T^6x_0) \\ &\leq a_1 \frac{\lambda}{\beta} v + (a_2 + b_1) \left(\frac{\lambda}{\beta}\right)^2 v + b_2\, d(T^5x_0, T^6x_0) \\ &= \left(\frac{\lambda}{\beta}\right)^2 \left[a_1 \left(\frac{\beta}{\lambda}\right) v + (a_2 + b_1) v + \left(\frac{\beta}{\lambda}\right)^2 b_2\, d(T^5x_0, T^6x_0) \right] \\ &\leq \left(\frac{\lambda}{\beta}\right)^2 \left[a_1 \left(\frac{\beta}{\lambda}\right) v + \left(\frac{\beta}{\lambda}\right) (a_2 + b_1) v + \left(\frac{\beta}{\lambda}\right)^2 b_2\, d(T^5x_0, T^6x_0) \right] \\ &= \left(\frac{\lambda}{\beta}\right)^2 \left[\left(\frac{\beta}{\lambda}\right) (a_1 + a_2 + b_1) v + \left(\frac{\beta}{\lambda}\right)^2 b_2\, d(T^5x_0, T^6x_0) \right] \\ &\leq \left(\frac{1}{\beta}\right)^2 \left[\left(\frac{\beta}{\lambda}\right) (a_1 + a_2 + b_1)^3 v + (\beta)^2 b_2\, d(T^5x_0, T^6x_0) \right] \end{aligned}$$

which implies $(\frac{\lambda}{\beta})\beta^3 d(T^5x_0, T^6x_0) \leq (1 - b_2)^3 d(T^5x_0, T^6x_0) \leq (\lambda)^3 v$. Hence,

$$d(T^5x_0, T^6x_0) \leq \left(\frac{\lambda}{\beta}\right)^2 v.$$

By continuing this process and using a technique similar to that in the proof of Theorem 4 of [39], it is easy to see that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, there exists $u \in X$ such that $x_n \to u$. Since T is orbitally continuous, $Tx_n \to Tu$ and so $Tu = u$. Now, we show that T has a unique fixed point. Let y and z be fixed points of T. Then,

$$d(y, z) = d(T^2y, T^2z) \leq a_1\, d(y, Ty) + a_2\, d(Ty, T^2y) + b_1\, d(z, Tz) + b_2\, d(Tz, T^2z)$$

and so $y = z$. Now, we prove that $F(T) = F(T^2)$. Let $y \in F(T^2)$. Then, we have

$$\begin{aligned} d(Ty, y) &= d(T^2Ty, T^2y) \leq a_1\, d(Ty, T^2y) + a_2\, d(T^2y, T^3y) + b_1\, d(y, Ty) + b_2\, d(Ty, T^2y) \\ &= (a_1 + a_2 + b_1 + b_2) d(y, Ty). \end{aligned}$$

Since $a + b < 1$, we get $Ty = y$. This completes the proof. □

Again, like in Examples 2.1, 2.2 and 2.2, we can show that there are nonlinear and discontinuous mappings satisfying the condition of Theorem 2.5. The following examples show that there are some mappings satisfying the condition of Theorem 2.5 while not satisfying the condition of Theorem 2.3 of [40].

Example 2.9. Let $X = \{1, 3, 5\}$, $d(x, y) = |x - y|$, $\leq = \{(1, 1), (3, 3), (5, 5)\}$ and T be a self-map on X defined by $T1 = 3$, $T3 = 1$ and $T5 = 5$. Then, on putting $x_0 = 5$, $a_1 = a_2 = b_1 = b_2 = \frac{1}{8}$, it is easy to see that T satisfies the condition of Theorem 2.5 while for $x = 1$ and $y = 3$ we have

$$2 = d(T^21, T^23) > a_1 d(1, T1) + a_2 d(T1, T^21) + b_1 d(3, T3) + b_2 d(T3, T^23) = 0.8.$$

This implies that T does not satisfy the condition of Theorem 2.3 of [40].

Example 2.10. Let $X = \{1, 2, 3, 4, 5\}$, $\leq = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 3)\}$, $d(x, y) = |x - y|$ and T be a self-map on X defined by $T1 = 4$, $T2 = 3$, $T3 = 3$, $T4 = 4$ and $T5 = 5$. Then, on putting $x_0 = 4$, $a_1 = \frac{1}{2}$, $a_2 = b_1 = b_2 = \frac{1}{10}$ it is easy to see that T satisfies the condition of Theorem 2.5 while for $x = 2$ and $y = 5$ we have $2 = d(3, 5) > a_1 d(2, 3) + a_2 d(3, 3) + b_1 d(5, 5) + b_2 d(5, 5) = 0.5$. This implies that T does not satisfy the condition of Theorem 2.3 of [40].

Let (X, d) be a metric space and T be a self-map on X which satisfies the contractive condition $d(Tx, Ty) \leq \lambda(d(x, Tx) + d(y, Ty))$ for all $x, y \in X$, where $\lambda \in [0, \frac{1}{2})$ is a constant (see [42]). Note that T is a convex contraction of order 2.

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References

[1] J.S. Vandergraft, Newton's method for convex operators in partially ordered spaces, SIAM Journal on Numerical Analysis 4 (3) (1967) 406–432.
[2] E.S. Wolk, Continuous convergence in partially ordered sets, General Topology and its Applications 5 (1975) 221–234.
[3] J. Sun, Y. Sun, A general principle on ordered sets and its applications to fixed point theory, Applicable Analysis 34 (1) (1989) 129–137.
[4] R.P. Agarwal, M.A. El-Gebeily, D. O'Regan, Generalized contractions in partially ordered metric spaces, Applicable Analysis 87 (1) (2008) 109–116.
[5] G. Wanka, ε-optimality to approximation in partially ordered spaces, Optimization 38 (1996) 1–18.
[6] I. Altun, B. Damjanovic, D. Djoric, Fixed point and common fixed point theorems on ordered cone metric spaces, Applied Mathematics Letters 23 (2010) 310–316.
[7] Z. Kadelburg, M. Pavlovic, S. Radenovic, Common fixed point theorems for ordered contractions and quasicontractions in ordered cone metric spaces, Computers and Mathematics with Applications 59 (2010) 3148–3159.
[8] M. Abbas, T. Nazir, Stojan Radenovic, Common fixed points of four maps in partially ordered metric spaces, Applied Mathematics Letters 24 (2011) 1520–1526.
[9] H. Aydi, B. Damjanovic, B. Samet, W. Shatanawi, Coupled fixed point theorems for nonlinear contractions in partially ordered G-metric spaces, Mathematical and Computer Modelling 54 (2011) 2443–2450.
[10] H. Aydi, H.K. Nashine, B. Samet, H. Yazidi, Coincidence and common fixed point results in partially ordered cone metric spaces and applications to integral equations, Nonlinear Analysis 74 (2011) 6814–6825.
[11] V. Berinde, Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces, Nonlinear Analysis 74 (2011) 7347–7355.
[12] V. Berinde, M. Borcut, Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces, Nonlinear Analysis 74 (2011) 4889–4897.
[13] M. Borcut, V. Berinde, Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces, Applied Mathematics and Computation 218 (2011) 5929–5936.
[14] M. Derafshpour, Sh. Rezapour, N. Shahzad, Best proximity points of cyclic φ-contractions in ordered metric spaces, Topological Methods in Nonlinear Analysis 37 (2011) 193–202.
[15] H.S. Ding, L. Lu, Coupled fixed point theorems in partially ordered cone metric spaces, Filomat 25 (2) (2011) 137–149.
[16] Wei-Shih Du, Coupled fixed point theorems for nonlinear contractions satisfied Mizoguchi–Takahashi's condition in quasi-ordered metric spaces, Fixed Point Theory and Applications (2010) Article ID 876372, 9 pages.
[17] Xin-qi Hu, Xiao-yan Ma, Coupled coincidence point theorems under contractive conditions in partially ordered probabilistic metric spaces, Nonlinear Analysis 74 (2011) 6451–6458.
[18] J. Jachymski, Equivalent conditions for generalized contractions on (ordered) metric spaces, Nonlinear Analysis 74 (2011) 768–774.
[19] N.V. Luong, N.X. Thuan, Coupled fixed points in partially ordered metric spaces and application, Nonlinear Analysis 74 (2011) 983–992.
[20] N.V. Luong, N.X. Thuan, Coupled fixed point theorems in partially ordered G-metric spaces, Mathematical and Computer Modelling 55 (2011) 1601–1609.
[21] H.K. Nashine, B. Samet, J.K. Kim, Fixed point results for contractions involving generalized altering distances in ordered metric spaces, Fixed Point Theory and Applications 2011 (5) (2011) 16.
[22] H.K. Nashine, B. Samet, C. Vetro, Monotone generalized nonlinear contractions and fixed point theorems in ordered metric spaces, Mathematical and Computer Modelling 54 (2011) 712–720.
[23] H.K. Nashine, B. Samet, Fixed point results for mappings satisfying (ψ, φ)-weakly contractive condition in partially ordered metric spaces, Nonlinear Analysis 74 (2011) 2201–2209.
[24] H.K. Nashine, W. Shatanawi, Coupled common fixed point theorems for a pair of commuting mappings in partially ordered complete metric spaces, Computers and Mathematics with Applications 62 (2011) 1984–1993.
[25] D. O'Regan, A. Petrusel, Fixed point theorems for generalized contractions in ordered metric spaces, Journal of Mathematical Analysis and Applications 341 (2008) 1241–1252.
[26] M.D. Rus, Fixed point theorems for generalized contractions in partially ordered metric spaces with semi-monotone metric, Nonlinear Analysis 74 (2011) 1804–1813.
[27] B. Samet, C. Vetro, Coupled fixed point theorems for multi-valued nonlinear contraction mappings in partially ordered metric spaces, Nonlinear Analysis 74 (2011) 4260–4268.
[28] W. Shatanawi, Some coincidence point results in cone metric spaces, Mathematical and Computer Modelling (2011), in press (doi:10.1016/j.mcm.2011.11.061).
[29] W. Shatanawi, On w-compatible mappings and common coupled coincidence point in cone metric spaces, Applied Mathematics Letters (2011), in press (doi:10.1016/j.aml.2011.10.037).
[30] W. Shatanawi, B. Samet, On (ψ, φ)-weakly contractive condition in partially ordered metric spaces, Computers and Mathematics with Applications 62 (2011) 3204–3214.
[31] D. Turkoglu, D. Binbashioglu, Some fixed point theorems for multivalued monotone mappings in ordered uniform spaces, Fixed Point Theory and Applications (2011) Article ID 186237, 12 pages.
[32] X. Zhang, Fixed point theorems of multivalued monotone mappings in ordered metric spaces, Applied Mathematics Letters 23 (2010) 235–240.
[33] Z. Li, Fixed point theorems in partially ordered complete metric spaces, Mathematical and Computer Modelling 54 (2011) 69–72.
[34] S.G. Jeong, B.E. Rhoades, Maps for which $F(T) = F(T^n)$, in: Fixed Point Theory and Applications, vol. 6, Nova Sci. Publ., New York, 2007, pp. 71–105.
[35] S.G. Jeong, B.E. Rhoades, More maps for which $F(T) = F(T^n)$, Demonstratio Mathematica 40 (3) (2007) 671–680.
[36] B.E. Rhoades, Some maps for which periodic and fixed points coincide, Fixed Point Theory and Applications 4 (2) (2003) 173–176.
[37] Z. Kadelburg, S. Radenović, V. Rakočević, Remarks on "Quasi-contraction on a cone metric space", Applied Mathematics Letters 22 (2009) 1674–1679.
[38] Sh. Rezapour, R.H. Haghi, N. Shahzad, Some notes on fixed points of quasi-contraction maps, Applied Mathematics Letters 23 (2010) 498–502.
[39] M.A. Alghamdi, Sh.H. Alnafei, S. Radenovic, N. Shahzad, Fixed point theorems for convex contraction mappings on cone metric spaces, Mathematical and Computer Modelling 54 (2011) 2020–2026.
[40] V.I. Istratescu, Some fixed point theorems for convex contraction mappings and mappings with convex diminishing diameters—I, Annali di Matematica Pura ed Applicata 130 (1982) 89–104.
[41] R.H. Haghi, Sh. Rezapour, N. Shahzad, Some fixed point generalizations are not real generalizations, Nonlinear Analysis 74 (2011) 1799–1803.
[42] Sh. Rezapour, R.H. Haghi, Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings", Journal of Mathematical Analysis and Applications 345 (2008) 719–724.